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TWO-DIMENSIONAL QUASI-STATIONARY TEMPERATURE DISTRIBUTION IN A MOVING INFINITE SLAB WITH ORTHOTROPIC PROPERTIES

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Abstract—An analytical solution is developed for the above problem with surface boundary conditions of distributed heat flux and convective loss. The case of a square-pulse heat-source on one surface with convective losses on both surfaces is considered in detail; temperature distributions are presented for a limited number of examples and an approximate solution is derived. The conduction/convection parameter, $\chi = \alpha/(Vl)$ is shown to provide criteria for neglecting energy transport due to conduction or convection in the direction of motion. For $\chi \leqslant 0.01$ only convection is important. For $0.01 \leqslant \chi \leqslant 10$ both conduction and convection are important. For $10 \leqslant \chi$ only conduction is important.

NOMENCLATURE

a, reference length;
B, Biot number;
c, specific heat [J/kg deg];
D, particle derivative defined in equation (14);
f, function defined in equation (37);
g, function defined in equation (31);
h, convective heat-transfer coefficient, [W/m^2 deg];
i, imaginary number, $\sqrt{-1}$;
I, integral;
k, thermal conductivity [W/m deg];
K, kernel of integral transform;
l, length of heat zone [m];
M, constant defined by equations (62–63);
n, indexing variable;
N, large integer defined by equation (47);
q, dimensionless heat flux, equation (17);
q'', heat flux [W/m^2];
t, time [s];
T, temperature [deg];
V, velocity [m/s];
w, part thickness [m];
x, coordinate parallel to direction of motion [m];
y, coordinate normal to direction of motion [m].

Greek symbols

α, thermal diffusivity [m^2/s];
β, Fourier transform variable, equation (27);
Δ, partial sum, equation (46);
ε, arbitrary small number;
ζ, dimensionless coordinate normal to direction of motion, equation (7);
η, dimensionless coordinate parallel to direction of motion, equation (6);
λ, dummy variable;
μ, heat storage parameter, equation (61);
ν, eigenvalue, equation (23);
τ, Fourier number or dimensionless time, equation (8);
π, irrational number;
ρ, density [kg/m^3];
ϕ, dimensionless temperature, equation (5);
Φ, dimensionless temperature in approximate solutions;
Φ'', renormalized dimensionless temperature, equation (64);
χ, conduction/convection parameter, equation (12);
ψ, dimensionless velocity, equation (9).

Subscripts

0, surroundings;
1, lower surface, $\zeta = 0$;
2, upper surface, $\zeta = 1$;
a, average;
c, characteristic;
i, indexing integer;
max, maximum;
n, indexing integer;
x, x direction;
y, y direction.

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Superscripts

- $^{-}$, finite Fourier integral transform, equation (20);
- $'$, dummy variable;
- $^{=}$, Fourier integral transform, equation (27).

INTRODUCTION

THE PRESENT work considers a body moving at a constant velocity relative to a steady heat source. If the body is of infinite extent in the direction of motion (i.e. all surfaces are parallel to the direction of motion), then a steady state temperature distribution will result when viewed in a coordinate system fixed to the heat source. This class of problems has been denoted quasi-stationary heat conduction [1] and has been studied by a number of investigators [1–8]. The engineering applications for these solutions are primarily in process engineering and include welding [1, 3], continuous extrusion [1], continuous quenching [1], lathe turning [3], grinding [5, 8], laser scribing, wave soldering, and other conveyorized thermal processes.

Rosenthal, in his excellent paper [1], surveyed prior work and presented a number of new solutions to quasi-stationary problems. These solutions included both singular and finite sources and infinite, semi-infinite, finite, and thin bodies. In particular he considered the temperature distribution in an insulated body of finite thickness ("slab") due to a square-pulse heat source.

Carslaw and Jaeger [2] summarized the solutions in the literature through 1958. These solutions included the insulated semi-infinite body with line and square-pulse heat-sources, thin body with line source and convective losses, and the slab problem of Rosenthal [1].

Watts [4] discussed the quasi-steady temperature distribution in solid and hollow cylinders subjected to ring heat sources. He included surface heat loss due to convection. DesRusseaux and Zerkle [5] presented the semi-infinite body with square-pulse source and surface losses.

Crisp [8] in his thesis considered a number of moving bodies of finite length. Because of the finite length, the solution was transient rather than quasi-stationary. His solutions included the insulated finite slab subject to a square-pulse heat source. Other transient solutions for bodies of finite length are discussed in Carslaw and Jaeger ([2], pp. 387–391). Cobble [9] considered a thin finite body subject to a discretely moving point source, and he included surface losses in his analysis.

References [6] and [7] considered a special sub-class of the quasi-stationary problem. Both references assumed that the part velocity was sufficiently great that thermal conduction in the direction of motion was negligible compared to thermal convection due to part motion. Ling and Yang [6] considered the insulated semi-infinite body with distributed surface heating. Both papers presented limited justification for neglecting conduction.

The present paper determines the quasi-steady temperature distribution in an infinite moving slab subjected to distributed heat sources and convective losses on both surfaces. The paper considers in detail the case of a square-pulse heat-source on one surface with convective losses on both surfaces.

The motivation for the paper was to determine criteria for neglecting conduction and convection in the direction of motion. Such criteria permit the reduction of the two-dimensional quasi-stationary problem to transient one-dimensional conduction if conduction is negligible and to steady-state two-dimensional conduction if convection is negligible. These reductions permit solution of many quasi-stationary problems by utilizing the numerous solutions available in traditional references [2].

In the present work, appropriate dimensionless variables are chosen and an analytical solution is obtained using Fourier transforms. The limiting form of the solution for negligible conduction in the traversing direction is presented. The general solution is then reduced to the square-pulse heat-source problem. An approximate solution is developed for this problem. The results of the approximate solution suggest modified dimensionless parameters which reduce the number of significant parameters. Graphical temperature distributions calculated from the exact solution are presented for a limited number of cases. General criteria are presented for neglecting conduction and convection in the direction of motion, and the range of validity of the approximate solution is determined.

MODEL

Figure 1 illustrates the geometry of the system. The heated part is of finite thickness, w, and infinite length. It moves with velocity, V, in the positive x direction. The top and bottom surfaces are heated by arbitrary heat fluxes, $q_i''(x)$, and are also convectively cooled. The thermal properties are temperature independent, but

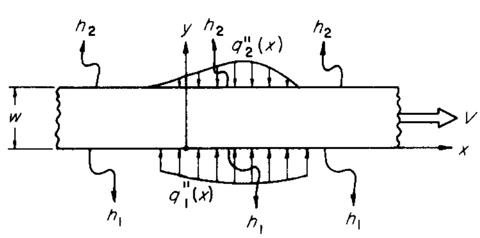

FIG. 1. Heat-transfer model.

the part is assumed orthotropic with different thermal conductivity in the x and y direction. The mathematical model for the temperature T is:

Differential equation

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = \rho c \frac{DT}{Dt}. \quad (1)$$

Boundary conditions

$$k_y \frac{\partial T}{\partial y} - h_1(T - T_0) = -q_1''(x); \; y = 0, \; -\infty \leqslant x \leqslant \infty \quad (2)$$

$$k_y \frac{\partial T}{\partial y} + h_2(T - T_0) = q_2''(x); \; y = w, \; -\infty \leqslant x \leqslant \infty \quad (3)$$

$$\frac{\partial T}{\partial x} \to 0, \; T \to T_0; \; x \to \pm\infty \qquad 0 \leqslant y \leqslant w. \quad (4)$$

Defining the following dimensionless parameters:

$$\phi = \frac{(T - T_0)k_y}{q_c'' w},$$

where q_c'' is a characteristic heat flux; (5)

$\eta = \frac{x}{l}$, where l is a characteristic length in the x direction; (6)

$$\zeta = y/w; \quad (7)$$

$$\tau = \frac{\alpha_y t}{w^2}, \text{ where } \alpha_y = k_y/\rho c; \quad (8)$$

$$\psi = \frac{Vw^2}{\alpha_y l} = \frac{w^2}{\alpha_y l}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}\tau}; \quad (9)$$

$$B_1 = h_1 w/k_y; \quad (10)$$

$$B_2 = h_2 w/k_y; \quad (11)$$

$$\chi = \alpha_x/(Vl), \quad \text{where} \quad \alpha_x = k_x/\rho c; \quad (12)$$

and substituting into the differential equations yields

$$\chi\psi \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2} = \frac{D\phi}{D\tau}. \quad (13)$$

The particle derivative, $D\phi/D\tau$, in an Eulerian coordinate system (a system fixed in space) is

$$\frac{D\phi(\eta, \zeta, \tau)}{D\tau} \equiv \frac{\partial \phi(\eta, \zeta, \tau)}{\partial \tau} + \frac{\partial \phi(\eta, \zeta, \tau)}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau}. \quad (14)$$

From equation (9)

$$\frac{D\phi(\eta, \zeta, \tau)}{D\tau} \equiv \frac{\partial \phi(\eta, \zeta, \tau)}{\partial \tau} + \psi \frac{\partial \phi(\eta, \zeta, \tau)}{\partial \eta}. \quad (15)$$

For a part moving at constant velocity, ψ, $\partial\phi(\eta, \zeta, \tau)/\partial\tau$ will be zero, and there will be a quasi-stationary temperature distribution. Thus, for constant velocity the particle derivative reduces to only the convective term $\psi\partial\phi/\partial\eta$ and the differential equation becomes

$$\psi\left(\chi \frac{\partial^2 \phi}{\partial \eta^2} - \frac{\partial \phi}{\partial \eta}\right) + \frac{\partial^2 \phi}{\partial \zeta^2} = 0. \quad (16)$$

The transformed boundary conditions are

$$\frac{\partial \phi}{\partial \zeta} - B_1\phi = -\frac{q_1''(x)}{q_c''} \equiv -q_1(\eta);$$
$$\zeta = 0, \; -\infty < \eta < \infty \quad (17)$$

$$\frac{\partial \phi}{\partial \zeta} + B_2\phi = \frac{q_2''(x)}{q_c''} \equiv q_2(\eta); \; \zeta = 1, \; -\infty < \eta < \infty \quad (18)$$

$$\frac{\partial \phi}{\partial \eta} \to 0, \; \phi \to 0; \; \eta \to \pm\infty, \; 0 \leqslant \zeta \leqslant 1. \quad (19)$$

The variable ϕ is the dimensionless temperature variable; it is normalized using a characteristic heat flux in the y direction. η and ζ are the dimensionless coordinate variables in the x and y directions, respectively; τ is the dimensionless time or Fourier number; it is defined in terms of a characteristic time for diffusion in the y direction. ψ is the dimensionless velocity and chosen to be $\mathrm{d}\eta/\mathrm{d}\tau$ to correspond to the definition of dimensional velocity, $V = \mathrm{d}x/\mathrm{d}t$. B_1 and B_2 are the Biot numbers for the bottom and top surfaces, respectively. χ is the ratio of conduction of energy to convection of energy in the x direction; this interpretation of χ is most clearly seen by observing the term in parenthesis in equation (16). χ is the parameter multiplying the conduction term in the η direction $\partial^2\phi/\partial\eta^2$. If $\chi \to 0$, it implies that this conduction term is negligible compared to the convection term $\partial\phi/\partial\eta$; if χ is of order one, it implies that both terms are of equal importance. If $\chi \to \infty$ it implies that convection is not important. While the definition of χ resembles a reciprocal Peclet number, it is preferable to make a distinction just as a Biot number is differentiated from a Nusselt number.

EXACT SOLUTION

The solution to the above model is obtained by first applying a finite Fourier integral transform in the ζ direction followed by a Fourier integral transform in the η direction. The use of transformations to solve the problem is dictated by the non-homogeneous boundary conditions in the ζ direction.

We define a finite Fourier integral transform

$$\bar{\phi}(\eta, v_n) \equiv \int_0^1 K(v_n, \zeta')\phi(\eta, \zeta')\mathrm{d}\zeta' \quad (20)$$

with inversion formula

$$\phi(\eta, \zeta) = \sum_{n=0}^{\infty} K(v_n, \zeta)\bar{\phi}(\eta, v_n). \quad (21)$$

The appropriate kernel, $K(v_n, \zeta)$, for the finite Fourier transform is the normalized eigenfunction of the auxiliary homogeneous eigenvalue problem (i.e. the problem in which $q_1(\eta)$ and $q_2(\eta) = 0$ for $-\infty \leqslant \eta \leqslant +\infty$)

where v_n is the appropriate eigenvalue [10]. For the present problem, the appropriate kernel is

$$K(v_n,\zeta)=2^{1/2}\frac{v_n\cos v_n\zeta+B_1\sin v_n\zeta}{\left[(v_n^2+B_1^2)\left(\dfrac{v_n^2+B_2^2+B_2}{v_n^2+B_2^2}\right)+B_1\right]^{1/2}} \quad (22)$$

and the eigenvalues are the positive roots of ([10], p. 50)

$$\tan v_n=\frac{v_n(B_1+B_2)}{v_n^2-B_1B_2}. \quad (23)$$

The integral transformation, equation (20), is applied to the differential equation, equation (16) yielding

$$\psi\left[\chi\frac{d^2\bar{\Phi}(\eta,v_n)}{d\eta^2}-\frac{d\bar{\Phi}}{d\eta}(\eta,v_n)\right]+\int_0^1 K(v_n,\zeta')\frac{\partial^2\phi(\eta,\zeta')d\zeta'}{\partial\zeta'^2}=0. \quad (24)$$

The integral term in equation (24) may be evaluated by integrating by parts twice. Performing the necessary algebra and utilizing equations (17), (18), (22) and (23) yields the following ordinary differential equation

$$\psi\left[\chi\frac{d^2\bar{\Phi}(\eta,v_n)}{d\eta^2}-\frac{d\bar{\Phi}}{d\eta}(\eta,v_n)\right]-v_n^2\bar{\Phi}(\eta,v_n)+K(v_n,1)q_2(\eta)+K(v_n,0)q_1(\eta)=0 \quad (25)$$

with transformed boundary conditions

$$\eta\to\pm\infty;\quad \frac{d\bar{\Phi}}{d\eta}\to 0,\ \bar{\Phi}\to 0. \quad (26)$$

We define a Fourier integral transform

$$\bar{\bar{\Phi}}(\beta,v_n)=\frac{1}{(2\pi)}1/2\int_{-\infty}^{\infty}\bar{\Phi}(\eta',v_n)e^{i\beta\eta'}d\eta'. \quad (27)$$

With inversion formula

$$\bar{\Phi}(\eta,v_n)=\frac{1}{(2\pi)}1/2\int_{-\infty}^{\infty}\bar{\bar{\Phi}}(\beta,v_n)e^{-i\beta\eta}d\beta. \quad (28)$$

Applying the Fourier transformation (equation (27)) to equation (25), and using the transformed boundary condition (equation (26)) yields

$$-\psi\chi\beta^2\bar{\bar{\Phi}}(\beta,v_n)+i\psi\beta\bar{\bar{\Phi}}(\beta,v_n)-v_n^2\bar{\bar{\Phi}}(\beta,v_n)+\frac{1}{(2\pi)^{1/2}}\times\int_{-\infty}^{\infty}[K(v_n,1)q_2(\lambda)+K(v_n,0)q_1(\lambda)]e^{i\beta\lambda}d\lambda=0. \quad (29)$$

It is assumed that the total heat input is finite so that the integral exists and is bounded.

Solving equation (29) for $\bar{\bar{\Phi}}(\beta,v_n)$ yields

$$\bar{\bar{\Phi}}(\beta,v_n)=\frac{\bar{g}(\beta)}{(\psi\chi\beta^2-i\psi\beta+v_n^2)}, \quad (30)$$

where

$$g(\eta)\equiv K(v_n,1)q_2(\eta)+K(v_n,0)q_1(\eta) \quad (31)$$

and the Fourier transformation, $\bar{g}$, of g is defined by equation (27).

Applying equation (28), the inversion formula, to equation (30) yields

$$\bar{\Phi}(\eta,v_n)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\bar{f}(\beta)\bar{g}(\beta)e^{-i\beta\eta}d\beta, \quad (32)$$

where

$$\bar{f}(\beta)\equiv\frac{1}{\psi\chi\beta^2-i\psi\beta+v_n^2}. \quad (33)$$

Applying the convolution theorem [11],

$$\int_{-\infty}^{\infty}\bar{f}(\beta)\bar{g}(\beta)e^{-i\beta\eta}d\beta=\int_{-\infty}^{\infty}g(\lambda)f(\eta-\lambda)d\lambda, \quad (34)$$

to equation (32) yields

$$\bar{\Phi}(\eta,v_n)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}[K(v_n,1)q_2(\lambda)+K(v_n,0)q_1(\lambda)]f(\eta-\lambda)d\lambda. \quad (35)$$

Applying the finite Fourier transformation inversion formula, equation (21) and equation (22), yields

$$\phi(\eta,\zeta)=\frac{2^{1/2}}{\pi}\sum_{n=1}^{\infty}\frac{(v_n\cos v_n\zeta+B_1\sin v_n)\displaystyle\int_{-\infty}^{\infty}[(v_n\cos v_n+B_1\sin v_n)q_2(\lambda)+v_nq_1(\lambda)]f(\eta-\lambda)d\lambda}{\left[\left(\dfrac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right]}. \quad (36)$$

$f(\eta)$, which by the inversion formula, equation (28), and equation (33) is

$$f(\eta)=\frac{1}{\psi\chi(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-i\beta\eta}d\beta}{\beta^2-\dfrac{i}{\chi}\beta+\dfrac{v_n^2}{\psi\chi}}, \quad (37)$$

is evaluated by integrating along a hemicircular contour in the complex plane using the residue theorem and noting that the two simple poles are located on the imaginary axis.

The solution for the temperature is

$$\phi(\eta,\zeta)=\frac{2}{\psi}\sum_{n=1}^{\infty}$$

$$\times\frac{(v_n\cos v_n\zeta+B_1\sin v_n\zeta)\displaystyle\int_{-\infty}^{\infty}[(v_n\cos v_n+B_1\sin v_n)q_2(\lambda)+v_nq_1(\lambda)]\exp\left[-\left|\frac{\eta-\lambda}{2\chi}\right|(1+4\chi v_n^2/\psi)^{1/2}+\frac{\eta-\lambda}{2\chi}\right]\mathrm{d}\lambda}{\left[\left(\dfrac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right](1+4\chi v_n^2/\psi)^{1/2}} \tag{38}$$

where v_n is defined by equation (23).

SOLUTION NEGLECTING CONDUCTION

When the dimensional velocity becomes sufficiently large, conduction in the η direction becomes insignificant compared to convection. The dimensionless variable which determines this condition is χ. A primary purpose of this work is to determine the maximum value of χ for which conduction can be neglected. The form of the solution for $\chi\to 0$ can be derived either by evaluating the limit of equation (38) or by transforming equation (25) to the Lagrangian coordinate system and utilizing Laplace transforms ([11], pp. 29–30) and the Faltung Theorem ([11], p. 31). The result is:

$$\phi(\tau,\zeta)=2\sum_{n=1}^{\infty}\frac{(v_n\cos v_n\zeta+B_1\sin v_n\zeta)\displaystyle\int_0^{\tau}[(v_n\cos v_n+B_1\sin v_n)q_2(\lambda')+v_nq_1(\lambda')]}{\left[\left(\dfrac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right]}\exp[-v_n^2(\tau-\lambda')\,\mathrm{d}\lambda', \tag{39}$$

where $\tau=0$ is chosen such that $q_1(\tau)=0$ for $\tau<0$. Equation (39) is a general solution to several specific transient problems whose solutions are given in ([2], numerous examples such as 3.11 iv, p. 125).

One advantage of doing a complete derivation starting from the differential equation [equation (13)] with the Lagrangian form of the total derivative is to demonstrate that equation (39) has broad validity. If the conduction/convection parameter, χ, is negligibly small, then the Lagrangian form of the differential equation is the usual one-dimensional transient heat-conduction equation independent of whether the velocity is constant or time dependent. Thus, equation (39) is valid for variable, but large (see the definition of χ) velocity.

APPLICATION OF MODEL TO SQUARE-PULSE HEAT SOURCE

In order to study the above model in greater detail, the model is now specialized to a specific problem of practical interest; namely, a process which involves convective cooling top and bottom and a square-pulse heat source of length l on the upper surface. Mathematically, the boundary conditions for this model are

$$q_1(\eta)=0;\qquad -\infty\leqslant\eta\leqslant\infty \tag{40}$$

$$q_2(\eta)=0;\qquad \eta<0,\ \eta>1 \tag{41}$$

$$q_2(\eta)=1;\qquad 0\leqslant\eta\leqslant 1. \tag{42}$$

Substitution of these boundary conditions into equation (38) and evaluation of the resulting integrals yields the temperature distribution

$$\phi(\eta,\zeta)=4\frac{\chi}{\psi}\sum_{n=1}^{\infty}$$

$$\times\frac{(v_n\cos v_n+B_1\sin v_n)(v_n\cos v_n\zeta+B_1\sin v_n\zeta)\left[\exp\left\{-\dfrac{\eta}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}-1]\right\}\right]\left\{\exp\dfrac{1}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}-1]-1\right\}}{\left[\left(\dfrac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right](1+4\chi v_n^2/\psi)^{1/2}[(1+4\chi v_n^2/\psi)^{1/2}-1]},$$

$$\eta>1 \tag{43}$$

$$\phi(\eta,\zeta) = -\frac{4\chi}{\psi}\sum_{n=1}^{\infty}$$

$$\times \frac{(v_n\cos v_n + B_1\sin v_n)(v_n\cos v_n\zeta + B_1\sin v_n\zeta)\exp\left\{\frac{\eta}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}+1]\right\}\left[\exp\left\{\frac{-1}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}+1]\right\}-1\right]}{\left[\left(\frac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right](1+4\chi v_n^2/\psi)^{1/2}[(1+4\chi v_n^2/\psi)^{1/2}+1]},$$

$$\eta < 0 \quad (44)$$

$$\phi(\eta,\zeta) = -\sum_{n=1}^{\infty}\frac{(v_n\cos v_n + B_1\sin v_n)(v_n\cos v_n\zeta + B_1\sin v_n\zeta)}{\left[\left(\frac{v_n^2+B_1^2}{v_n^2+B_2^2}\right)(v_n^2+B_2^2+B_2)+B_1\right]v_n^2(1+4\chi v_n^2/\psi)^{1/2}}$$

$$\times\left\{[(1+4\chi v_n^2/\psi)^{1/2}+1]\exp\left\{\frac{-\eta}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}-1]\right\}\right.$$

$$\left.+[(1+4\chi v_n^2/\psi)^{1/2}-1]\exp\left\{\frac{\eta-1}{2\chi}[(1+4\chi v_n^2/\psi)^{1/2}+1]\right\}-2(1+4\chi v_n^2/\psi)^{1/2}\right\}, \quad 0 \leqslant \eta \leqslant 1 \quad (45)$$

EVALUATION OF EXACT SOLUTION

In order to use the analytical solution [equations (43–45)], a computer program has been written which computes the eigenvalues [equation (23)] and then evaluates the summations in the analytical solution. The eigenvalues are computed using Muller's iteration scheme of successive bisection and inverse parabolic interpolation [12]. The details of the procedure are discussed in the appendix.

For certain cases this series solution converges extremely slowly, requiring considerably more computer time than numerical techniques such as finite difference or finite element.

In the regions $\eta < 0$ and $\eta > 1$ the exponential coefficients insure that as $|\eta| \to \infty$ the succeeding sinusoidal terms in equations (43–44) rapidly go to zero. In the range $0 \leqslant \eta \leqslant 1$ as $\zeta \to 1$ the terms in the summation of equation (45) are not of alternating sign resulting in slow monotonic convergence. For the example of $\chi = 0{\cdot}0003506$, $\psi = 4{\cdot}5633$, $B_1 = 0{\cdot}0367$, and $B_2 = 94{\cdot}401$, the summation of the first 9600 terms at $\eta = 0{\cdot}8$, $\zeta = 1$ is over 0·2 per cent from the limiting value. Figure 2 illustrates convergence of the series for this example problem at $\eta = 0{\cdot}8$. The figure presents temperature distributions in the ζ direction which were calculated by truncating the series of equation (45) after 100, 200, 400, 1000 and 2000 terms. The series was evaluated at ζ increment of 0·01 and the points connected by straight lines. The results for the present example indicate that convergence is obtained for $\zeta < 0{\cdot}99$ using 200 terms. However, to obtain convergence at $\zeta = 1$, 2000 terms are barely sufficient. The behavior is particularly disturbing since the temperature of greatest interest is the surface temperature.

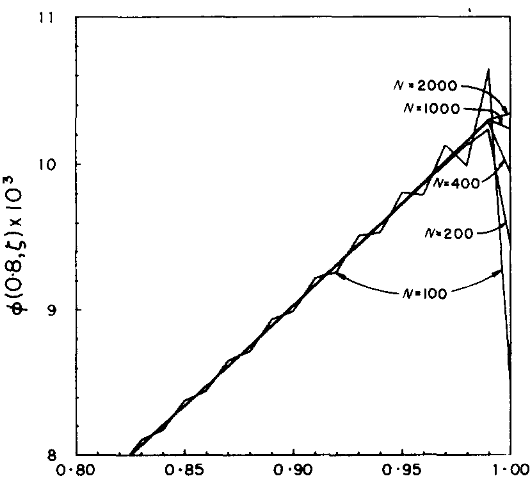

FIG. 2. Dimensionless temperature distribution across slab as a function of the number of terms in the summation.

Convergence can be improved for $\zeta = 1$, $0 \leqslant \eta \leqslant 1$, by determining the partial sum of the terms for $n \geqslant N$ where N is an arbitrary, large number. For $\zeta = 1$, using the definition of the eigenvalue, equation (23), and equation (45) the partial sum can be written as

$$\Delta\phi_N(\eta, 1) = -\sum_{n=N}^{\infty} \frac{\cos^2 \nu_n \left(1 + \dfrac{\nu_n(B_1+B_2)}{\nu_n^2 - B_1 B_2}\right)^2}{\left[\left(\dfrac{\nu_n^2+B_1^2}{\nu_n^2+B_2^2}\right)(\nu_n^2+B_2^2+B_2)+B_1\right](1+4\chi\nu_n^2/\psi)^{1/2}}$$
$$\times \left\{[(1+4\chi\nu_n^2/\psi)^{1/2}+1]\exp\left[-\frac{\eta}{2\chi}[(1+4\chi\nu_n^2/\psi)^{1/2}-1]\right]\right.$$
$$\left. + [(1+4\chi\nu_n^2/\psi)^{1/2}-1]\exp\left[\frac{\eta-1}{2\chi}[(1+4\chi\nu_n^2/\psi)^{1/2}+1]\right] - 2(1+4\chi\nu_n^2/\psi)^{1/2}\right\} \quad (46)$$

N is chosen so that

$$\nu_N \gg B_1$$
$$\nu_N \gg B_2$$
$$\nu_N \gg \left(\frac{\psi}{4\chi}\right)^{1/2} \quad (47)$$
$$\nu_N \gg \frac{(\chi\psi)^{1/2}}{\varepsilon},$$

where ε is an arbitrary small number.

For the range $\varepsilon < \eta < 1-\varepsilon$ dropping terms which because of the choice of N are of higher order, equation (46) becomes

$$\Delta\phi_N \cong 2\sum_{n=N}^{\infty} \frac{\cos^2 \nu_n}{\nu_n^2}. \quad (48)$$

For large n, since $\nu_n \cong n\pi$ and $\cos^2(\nu_n) \cong 1$,

$$\Delta\phi_N \cong \frac{2}{\pi^2}\sum_{n=N}^{\infty}\frac{1}{n^2}. \quad (49)$$

The summation can be approximated by an integral [13].

$$\sum_{n=N}^{\infty}\frac{1}{n^2} \cong \int_N^{\infty}\frac{1}{n^2}\,dn = \frac{1}{N}; \quad (50)$$

thus,

$$\Delta\phi_N \cong \frac{2}{\pi^2 N} \quad \text{for} \quad \varepsilon < \eta < 1-\varepsilon. \quad (51)$$

In a similar manner it can be shown that

$$\Delta\phi_N \cong \frac{1}{\pi^2 N} \quad \text{for} \quad \eta = 0 \quad \text{or} \quad 1. \quad (52)$$

Figure 3 illustrates for the example problem the percent error for the truncated summation and truncated summation plus integrated remainder as a function of the logarithm of the number of terms. Adding the integrated remainder to the truncated solution dramatically improves convergence. One hundred and fifty terms plus remainder provide as good a prediction as 10 000 terms without the remainder.

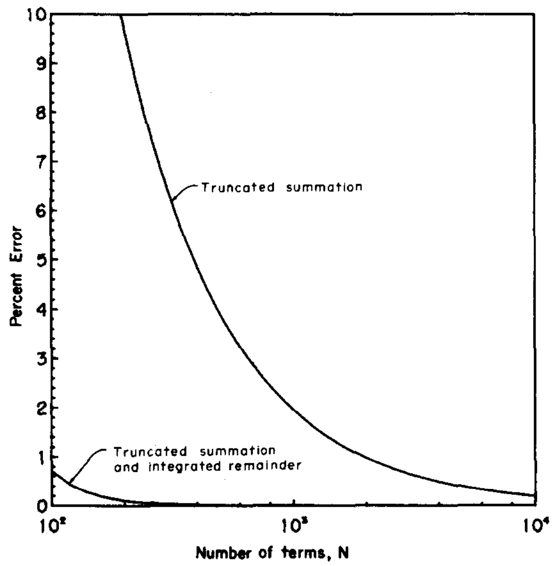

FIG. 3. Percent error of truncated summation and truncated plus remainder summation.

APPROXIMATE SOLUTION FOR SQUARE-PULSE HEAT SOURCE

Since the exact solution frequently does not converge rapidly, it is useful to obtain an approximate solution to the above problem which can be used for rapid calculations. As will be shown later, the approximate solution under certain circumstances reduces to the exact solution truncated after the first term.

The approximate solution is derived using an integral technique which allows the choice of function to represent the y dependency of temperature. A linear function has been chosen because it provides a realistic approximation to the exact profile for two cases of interest: (1) the isothermal distribution which occurs when $B_1 \ll 1$ and $B_2 \ll 1$, (2) the linear distribution which occurs when there is a heat sink at $y = 0$ such that $B_1 \gg 1$. The linear function is not a good choice for the case of an adiabatic surface at $y = 0$ ($B_2 > 1, B_1 = 0$); an obvious choice would be a parabolic distribution.

The approximate model is most readily developed in an Eulerian coordinate system in terms of the real variables. One performs an energy balance on a differential segment of the slab equating the increase in internal energy with the net heat flux to the differential volume.

$$k_x \frac{d^2T_a}{dx^2} - \rho c V \frac{dT_a}{dx} - \frac{h_1}{w}(T_1 - T_0) - \frac{h_2}{w}(T_2 - T_0) = 0; \quad x < 0, x > l \quad (53)$$

$$k_x \frac{d^2T_a}{dx^2} - \rho c V \frac{dT_a}{dx} + \frac{q_2''}{w} - \frac{h_1}{w}(T_1 - T_0) - \frac{h_2}{w}(T_2 - T_0); \quad 0 \leqslant x \leqslant l \quad (54)$$

$T_1(x)$ is the temperature at $y = 0$, $T_2(x)$ is the temperature at $y = w$, and T_a is the average temperature of the slab. For the assumed linear distribution

$$T(x, y) = T_1(x) + [T_2(x) - T_1(x)]\frac{y}{w}, \quad (55)$$

$$T_a(x) = \frac{T_1(x) + T_2(x)}{2}. \quad (56)$$

A relationship between $T_1(x)$ and $T_2(x)$ is obtained by requiring T to satisfy the differential boundary condition at one of the surfaces. The boundary condition on the lower surface is chosen [equation (2)] in order to accurately model the case of a heat sink on this surface ($B_1 \gg 1$). In terms of the present problem with its assumed linear distribution, equation (2) becomes

$$k_y \frac{(T_2 - T_1)}{w} - h_1(T_1 - T_0) = 0; \quad y = 0, \ -\infty \leqslant x \leqslant \infty. \quad (57)$$

Substituting equations (56) and (57) into (53) and (54), integrating the three differential equations, and applying the appropriate six boundary conditions yields in terms of dimensionless variables*

$$\Phi''(\eta) = -\eta_{max} \frac{M_2}{M_1} \frac{1 - \exp(-M_1)}{1 - \exp(M_2 \eta_{max})} \exp(M_1 \eta), \quad \eta < 0, \quad (58)$$

$$\Phi''(\eta) = \frac{1 - \eta_{max}[\exp(M_2\eta) - \frac{M_2}{M_1}\exp(M_1(\eta - 1))]}{1 - \exp(M_2\eta_{max})}, \quad 0 \leqslant \eta \leqslant 1, \quad (59)$$

*The variable Φ is used to denote the dimensionless temperature in the approximate solution, ϕ is reserved to denote the exact solution.

$$\Phi''(\eta) = \eta_{max} \frac{[1 - \exp(M_2)]\exp(M_2(\eta - 1))}{1 - \exp(M_2\eta_{max})}, \quad \eta > 1; \quad (60)$$

where

$$\mu \equiv \frac{2}{\psi} \frac{B_1 + B_2 + B_1 B_2}{2 + B_1}, \quad (61)$$

$$M_1 \equiv \frac{1}{2\chi}[1 + (1 + 4\mu\chi)^{1/2}], \quad (62)$$

$$M_2 \equiv \frac{1}{2\chi}[1 - (1 + 4\mu\chi)^{1/2}], \quad (63)$$

$$\Phi''(\eta) \equiv \frac{B_1 + B_2 + B_1 B_2 \Phi(\eta, \zeta)}{(1 + B_1\zeta)\{1 - \exp[\mu/(1 + 4\mu\chi)^{1/2}]\}}, \quad (64)$$

$$\eta_{max} = \frac{M_1}{M_1 - M_2} = \frac{1 + (1 + 4\mu\chi)^{1/2}}{2(1 + 4\mu\chi)^{1/2}}. \quad (65)$$

η_{max} is the value of η at which the temperature, equation (59), is a maximum. For $\chi = 0$ the maximum value of dimensionless temperature will occur at $\eta = 1$ while for no motion of the slab ($\chi \to \infty$, $\psi = 0$) the maximum will occur at $\eta = \frac{1}{2}$. Thus, for all χ the point of maximum temperature, η_{max}, will be in the range $0{\cdot}5 \leqslant \eta_{max} \leqslant 1{\cdot}0$. The dimensionless temperature variable has been redefined in equation (64) so that it is independent of ζ and scaled so that its maximum value at $\eta = \eta_{max}$ is unity.

The above solution can be further simplified when $4\mu\chi \ll 1$. Applying this inequality to equations (58–60) yields

$$\lim_{4\mu\chi \to 0} \Phi''(\eta) = 0, \quad \eta < 0 \quad (66)$$

$$\lim_{4\mu\chi \to 0} \Phi''(\eta) = \frac{1 - \exp(-\mu\eta)}{1 - \exp(-\mu)}, \quad 0 \leqslant \eta \leqslant 1 \quad (67)$$

$$\lim_{4\mu\chi \to 0} \Phi''(\eta) = \exp[-\mu(\eta - 1)], \quad \eta > 1. \quad (68)$$

This limiting solution represents the appropriate solution for negligible conduction and is independent of χ. For small $\mu\chi$ and the assumed linear profile, there is no preheating by conduction for $\eta < 0$, and the thermal process consists of an exponential heating process from $0 \leqslant \eta \leqslant 1$ with a thermal time constant of $1/\mu$ followed by exponential cooling with the same time constant. Thus, the parameter μ is a reciprocal thermal time constant or energy storage parameter. Since μ equals a function of the Biot numbers divided by the dimensionless velocity ψ, it can also be considered a reciprocal velocity.

The above discussion suggests that the important parameter in addition to χ which characterizes the renormalized dimensionless temperature distribution

$\phi''(\eta, \zeta)$ is the energy storage parameter μ. It readily can be shown that

$$\mu_{\lim B_1 \to 0,\, B_2 \to 0} = \frac{B_1+B_2+B_1B_2}{\psi} = v_1^2/\psi. \quad (69)$$

Thus for small Biot number processes the appropriate solution is equivalent to the exact solution truncated after the first term except that the cosine and sine functions in the ζ direction have been replaced by a linear function.

RESULTS FOR A SQUARE-PULSE HEAT SOURCE

A series of parametric graphs have been prepared to aid in interpreting important physical regimes of the model. Although the results are based on the exact solution, the modified dimensionless temperature ϕ'' has been found to be more appropriate for presenting the results.

We first study the dependency of the temperature distribution at the heated surface on the conduction/convection parameter. Figure 4 graphs $\phi''(\eta, 1)$ vs η for

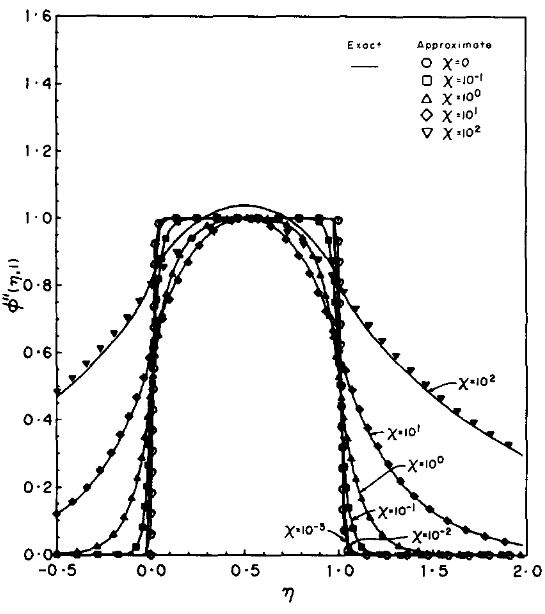

FIG. 4. Temperature distribution on heated surface: $\psi = 10^{-3}$ ($\mu = 100$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

various values of χ from 10^2 to 10^{-3}. The value of the remaining parameters are $B_1 = 0$, $B_2 = 0{\cdot}1$, and $\psi = 10^{-3}(\mu = 100)$; the values of B_1 and B_2 were arbitrarily selected as being typical of many manufacturing processes. For $\chi \leqslant 10^{-2}$ there is no preheating prior to $\eta = 0$; hence, conduction is negligible. The symmetric temperature distributions with significant preheating for $\chi = 10$ and 10^2 represent cases with negligible effect from part convection.

Figures 5–6 are similar to Fig. 4 except that the parameter ψ is increased two decades for each succeeding figure (μ is decreased two decades). On all figures we note that for $\chi \leqslant 10^{-2}$ there is no preheating due to conduction. We also note that the temperature distribution for the $\chi = 10^{-2}$ and 10^{-3} case are identical and hence, for $\chi \leqslant 10^{-2}$ the distributions are independent of χ. The solution for $\chi = 10^{-3}$ changes gradually from a symmetric square-pulse to an exponential rise followed by an exponential decay as ψ increases from 10^{-3} to 10.

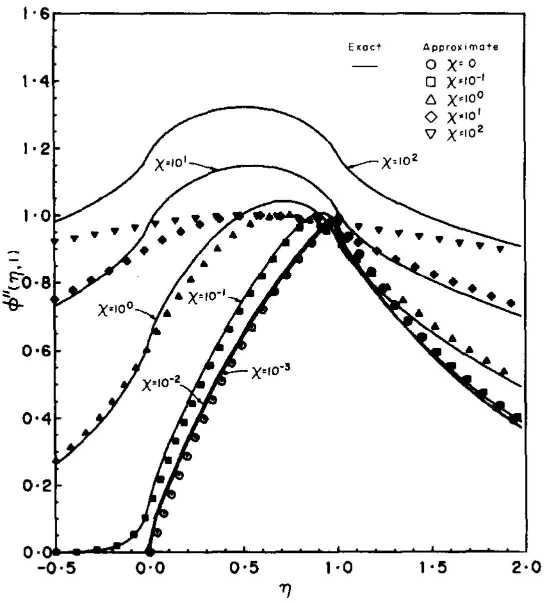

FIG. 5. Temperature distribution on heated surface: $\psi = 10^{-1}$ ($\mu = 1$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

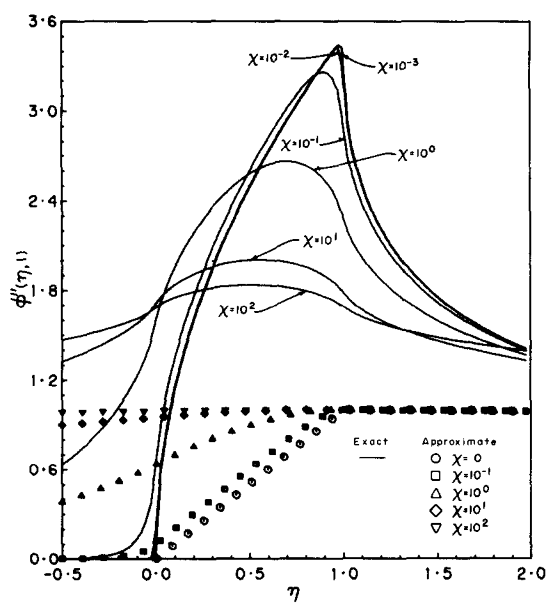

FIG. 6. Temperature distribution on heated surface: $\psi = 10$ ($\mu = 0{\cdot}01$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

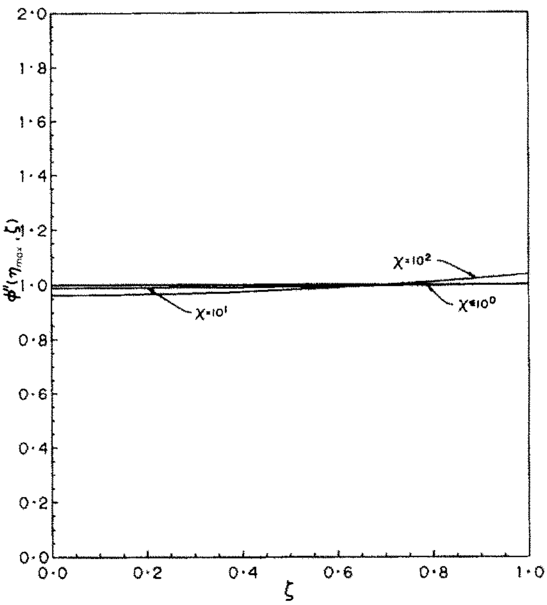

FIG. 7. Temperature distribution across part: $\psi = 10^{-3}$ ($\mu = 100$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

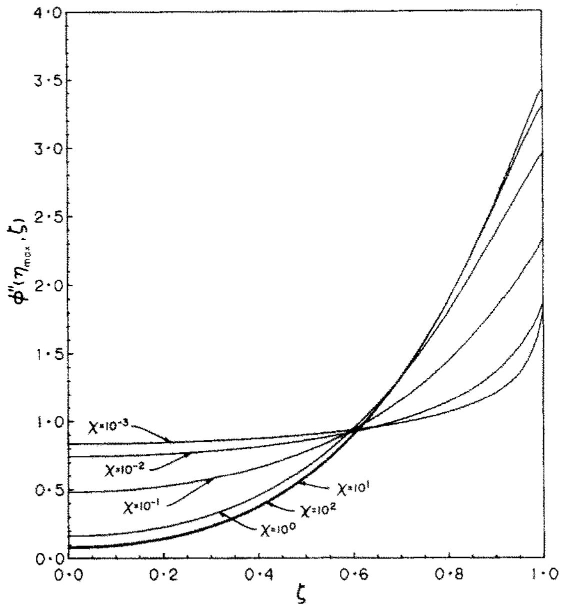

FIG. 9. Temperature distribution across part: $\psi = 10$ ($\mu = 0{\cdot}01$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

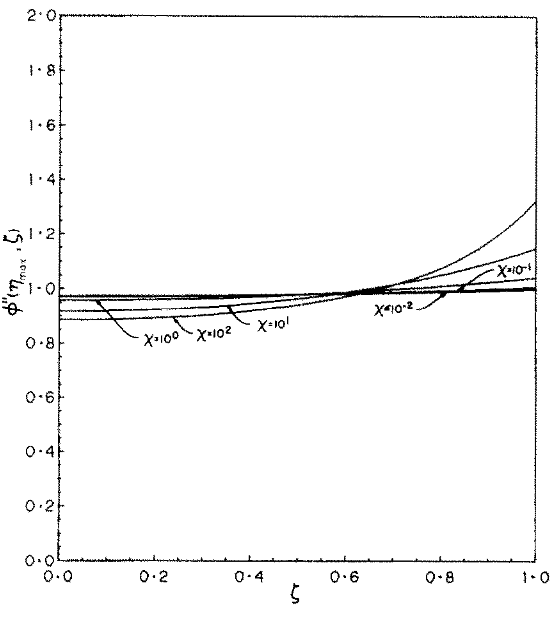

FIG. 8. Temperature distribution across part: $\psi = 10^{-1}$ ($\mu = 1$), $B_1 = 0$, $B_2 = 0{\cdot}1$.

Figures 7–9 are similar to Figs. 4–6 except that they plot the temperature distribution at η_{max} as a function of the coordinate ζ. The figures are again for $B_1 = 0$ and $B_2 = 0{\cdot}1$. Figure 7 is for $\psi = 10^{-3}(\mu = 100)$ and shows that $\phi''(\eta_{max}, \zeta)$ is independent of ζ for $\chi < 10$. Figures 8 and 9 show that as ψ increases $\phi''(\eta_{max}, \zeta)$ becomes a strong function of ζ. This functional dependency on ζ indicates that there is insufficient time for the effect of the heat source to diffuse through the part in the ζ direction in spite of the low Biot numbers which for Newtonian cooling would insure an isothermal part [14].

The preceding figures have presented η and ζ temperature distributions for various values of the parameters χ and ψ. It was shown that conduction in the direction of motion had a negligible effect on the on the surface temperature distributions when $\chi \leqslant 10^{-2}$. Figure 10 presents a quantitative analysis for determining when conduction in the x direction can be neglected for a wide range of μ, B_1 and B_2 values. An arbitrary criterion for neglecting conduction is that $\phi''(-0{\cdot}01, 1)/\phi''(\eta_{max}, 1) \leqslant 0{\cdot}05$. The criterion states that the surface temperature rise immediately preceding the heat zone should be less than 5 per cent of the surface temperature at η_{max}. The point $(0, 1)$ is not chosen since the heat flux is singular at this point. Curves were generated using Muller's iteration scheme [12] to determine those values of χ which satisfied the 5 per cent error criteria. These curves are plotted in Fig. 10 as a function μ and χ for the five cases $B_1 = 0$, $B_2 = 0{\cdot}1$; $B_1 = 0$, $B_2 = 1$; $B_1 = 0{\cdot}1$, $B_2 = 0$; $B_1 = 1{\cdot}0$, $B_2 = 0$; and $B_1 = 10$, $B_2 = 0$. They are the vertical lines slightly to the right of $\chi = 10^{-2}$. The minor bends in the curves may be caused by using $\phi''(\eta_{max}, 1)$, the exact temperature at the approximate maximum, rather than the actual maximum surface temperature. The results demonstrate that conduction

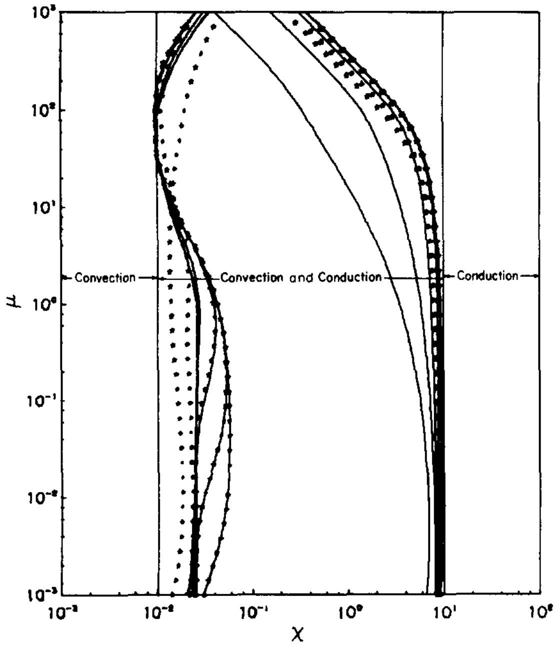

FIG. 10. Regions of negligible conduction and negligible convection.

in the direction of motion can be neglected for $\chi < 10^{-2}$ independent of the values of the other parameters, μ, B_1 and B_2.

A second series of curves on Fig. 10 show when convection due to part motion can be neglected. When convection is negligible, the temperature distribution in the part is symmetric, and, therefore, $\phi''(0{\cdot}5-\eta, \zeta) = \phi''(0{\cdot}5+\eta, \zeta)$. The criterion chosen to evaluate negligible convection is for the percent difference between the surface temperature at the end and beginning of the heater to be less than five per cent, or $[\phi''(1,1)-\phi''(0,1)]/\phi''(1,1) \leqslant 0{\cdot}05$. These curves again were generated using Muller's iteration scheme [12] and are the family of vertical lines which approach $\chi = 10$ as $\mu \to 0$. It is apparent from Fig. 10 that χ is also the unique dimensionless parameter which determines when convection can be neglected. For $\chi \geqslant 10$ convection is always negligible and only conduction must be considered. Only for $10^{-2} \leqslant \chi \leqslant 10$ must both conduction and convection be considered.

This author is unaware of any prior work presenting a criterion for neglecting part convection for quasi-stationary problems; however, in at least two other papers the authors have considered the question of criterion for neglecting conduction [6, 7]. Friedman and Lenz [7] consider a slab problem with prescribed temperature distributions on the two surfaces. For this very specific problem, the authors calculate a criterion for neglecting conduction. In terms of the notation of the present paper, they conclude that they can neglect conduction for $\chi \leqslant 0{\cdot}75 \times 10^{-2}$. Ling and Yang [6] in their paper discuss the effect of conduction on the heated surface temperature distribution. They state without supporting calculations that neglecting conduction has been shown to accurate for $\chi \leqslant 0{\cdot}1(2a/l)$ where a is an undefined reference length. They present a specific example problem involving a heat source with parabolic distribution for which $\chi = 1{\cdot}1 \times 10^{-3}$. Thus, the present analysis supports Ling and Yang's conclusion that conduction is unimportant for their example problem, but it does not corroborate their general criterion with its undefined parameter a.

VALIDITY OF APPROXIMATE SOLUTION

The preceding section has presented temperature distributions for a limited number of values of the parameters ψ, B_1 and B_2. It is impossible to present graphical results for all of the cases which may be of interest, therefore, the reader may need to calculate temperature distributions for examples not covered in the predecing section. The approximate solution is most appropriate for these calculations since it is readily evaluated on a pocket calculator while the exact solution requires lengthy summations on a digital computer. The purpose of this section is to illustrate the zone of validity for the approximate solution.

Comparison of exact surface temperature distributions with the approximate solutions in Figs. 4–6 shows that when the approximate solution predicts $\phi''(\eta_{max}, 1)$ it provides a good prediction of the entire surface temperature distribution. Because of the renormalization of the dimensionless temperature, the approximate solution always predicts that $\Phi''(\eta_{max}, 1) = 1$. One can examine Figs. 4–6 and determine for which cases the approximate solution is not valid. This technique is subjective and only a limited number of cases have been presented. Figure 11 is an objective presentation for a large number of cases. The criterion selected for determining the validity of the approximate solution is that the exact and approximate value for $\phi''(\eta_{max}, 1)$ differ by 5 per cent or less. Curves satisfying this criterion were generated using Muller's iteration scheme [17]. The results are presented as functions of η and χ with B_1 and B_2 as parameters. There are pairs of curves, one with $B_2 = 0$ and $B_1 = B$, the other with $B_1 = 0$ and $B_2 = B$, where $B = 10^{-3}$, 10^{-2}, 10^{-1}, 1, 10 and 100. If the parameter coordinates for a process place it to the upper left of the appropriate B curve, an approximate solution can be used to predict the maximum surface temperature within 5 per cent.

There are also two parametric curves on Fig. 11 for η_{max}. These curves are straight lines with a slope of -1 on the logarithmic coordinates of Fig. 11. Consider first the $\eta_{max} = 0{\cdot}52$ line. $\eta_{max} \to 0{\cdot}5$ implies a symmetric temperature distribution and consequently no convection. Thus, $\eta_{max} < 0{\cdot}52$ can be used as a criteria for neglecting convection in the approximate solution. Comparison with Fig. 10 reveals that this criterion is consistent with the results for the exact solution. Note that to the upper right of this line all of the temperature error curves have a slope of $+1$ indicating that for this region the zone of validity of the approximate solution is independent of μ/χ or velocity as expected. $\eta_{max} \to 1$ or $4\mu\chi \to 0$ implies no conduction. Thus, to the left of the $\eta_{max} = 0{\cdot}98 (\mu\chi = 0{\cdot}0213)$ line and above the appropriate B curve, equations (66–68) can be used to predict the temperature distribution. The horizontal B curves for $\eta_{max} > 0{\cdot}98$ indicate that the zone of validity of the approximate solution is essentially independent of the conduction parameter χ.

The triangular region $\mu\chi \geqslant 0{\cdot}0213$, $\chi \leqslant 10^{-2}$ at the upper left to Fig. 11 is a region for which conduction is negligible; but since $\mu\chi \geqslant 0{\cdot}0213$ the limiting form of the approximate solution has not been shown to be valid. Comparison of the general and limiting form of the approximate solution shows that for this region they are identical except in a small neighborhood of $\eta = 0$ and 1.

The range of validity of the approximate solution can be estimated for other values of B_1 and B_2 by calculating an effective Biot number, $B_e = 2(B_1+B_2+B_1B_2)/(2+B_1)$. The calculated B_e is then used with the effective Biot numbers shown on Fig. 11. This technique will give a conservative estimate of the range of validity of the approximate solution.

We next determine the region for which the approximate solution predicts the temperature distribution across the part at $\eta = \eta_{max}$. The approximate solution predicts that $\Phi''(\eta_{max}, \zeta) = 1$; however, since Φ is a linear function of ζ when Φ'' is constant, the region of validity for the approximate solution is not limited to negligible temperature gradients in ζ. In particular if the heated slab rests on a heat sink such that $h_1 \to \infty$ and hence $B_1 \gg 1$ then the actual temperature distribution will be essentially linear in the slab, the transformed dimensionless temperature, Φ'', will be constant, and the approximate solution, Φ'', will accurately model the linear temperature distribution across the slab. The approximate solution predicts that $\Phi''(\eta_{max}, 1) = \Phi''(\eta_{max}, 0) = 1$, whereas as shown by Figs. 7–9 $\phi''(\eta_{max}, 1) \geqslant 1$ and $\phi''(\eta_{max}, 0) < 1$ because of the finite rate of diffusion in the ζ direction. Consequently, a valid criteria for determining the zone of validity of the approximate solution is $\phi''(\eta_{max}, 1) - \phi''(\eta_{max}, 0) = 0{\cdot}05$.

Figure 12 presents curves satisfying this criteria. As in Fig. 11, they are plotted as a function of μ and χ with B_1 and B_2 as parameters. Again, curves are shown for η_{max} as a function of μ and χ and the previous comments apply. The results are very similar to the previous figure except for $B_1 = 0$, $B_2 > 1$. For these

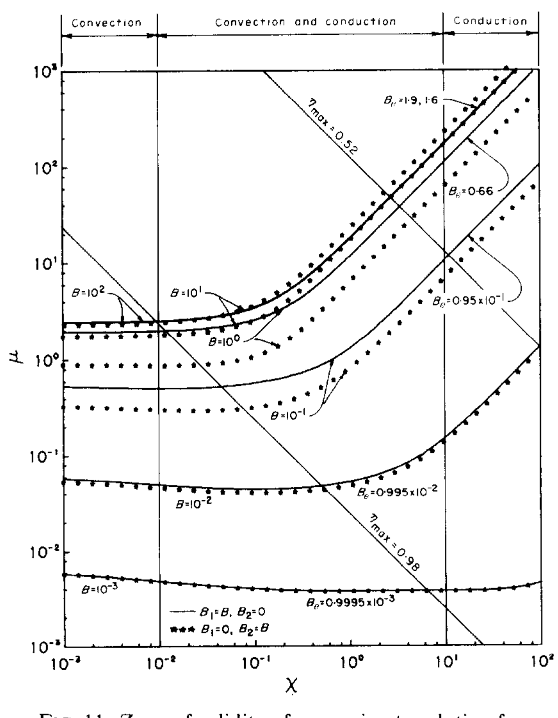

FIG. 11. Zone of validity of approximate solution for predicting $\phi''(\eta_{max}, 1)$.

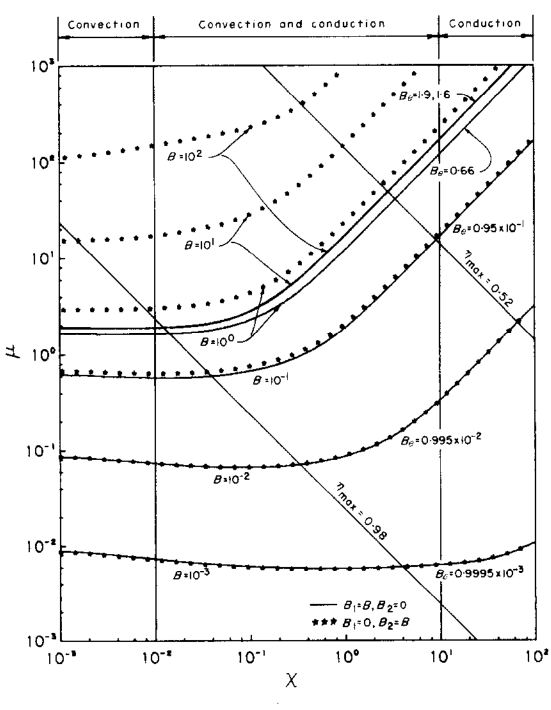

FIG. 12. Zone of negligible gradient for $\phi''(\eta_{max}, \zeta)$.

cases, the range of negligible gradient is a strong function of B_2, rapidly becoming smaller with increasing B_2. Hence, the approximate solution has only a small range of usefulness for $B_2 > 1$. For these cases the temperature distribution would be better approximated by a parabolic distribution with zero gradient at $\zeta = 0$. Conversely, for $B_1 > 1$, $B_2 = 0$ the range of negligible gradient becomes essentially independent of B_1. The large heat sink on the bottom surface produces a temperature profile linear in ζ.

SUMMARY

The present work has determined the temperature distribution in a two-dimensional quasi-stationary slab subjected to distributed heat sources and convective losses on both surfaces. Temperature distributions have been presented for the case of a square-pulse heat source on one surface with convective losses on both surfaces. Detailed analysis of this case has shown that conduction in the direction of motion has a negligible effect on energy transport for $\chi \leqslant 0{\cdot}01$. Conversely, for $\chi > 10$ convection due to part motion is negligible. Consequently, for $\chi \leqslant 0{\cdot}01$ the quasi-stationary two-dimensional problem reduces to transient one-dimensional conduction; whereas for $\chi \geqslant 10$ it becomes steady-state two-dimensional conduction.

Since numerical evaluation of the exact solution frequently requires extensive calculations, an approximate solution has been presented which can be evaluated readily on a pocket calculator. Figures are presented which delineate the zone of validity of this approximate solution as a function of the parameters of the system.

Acknowledgements—The author acknowledges many fruitful discussions with J. C. Mollendorf on the choice of dimensionless parameters and appropriate manner in which to present the graphical results. He further wishes to acknowledge the programming efforts of G. M. Wenger in computing the required data points and generating not only the numerous plots presented in this report, but innumerable others for these discussions.

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APPENDIX

Computation of Eigenvalues

In order to numerically evaluate the mathematical solution, one first calculates the eigenvalues which are the positive roots of equation (23) using Muller's iteration scheme [17]. In order to utilize Muller's scheme, one must calculate upper and lower bounds for each eigenvalue. Equation (23) is reformulated to the following form:

$$f(v_n) = v_n \tan v_n - \frac{v_n^2(B_1+B_2)}{v_n^2 - B_1 B_2} = 0. \tag{70}$$

Study of this function for all non-negative values of v_n, B_1 and B_2^* reveals that each root is located between two singular points; the left hand singularity being negative and the right hand singularity positive. Thus, to determine the appropriate lower and upper bonds for a given root, one must first determine the singularities.

One singularity will occur when the denominator in equation (70) is zero

$$v_n = (B_1 B_2)^{1/2}. \tag{71}$$

The second type of singularity will occur whenever the tangent is singular or for

$$v_n = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}. \tag{72}$$

*B_1 and B_2 may not both equal zero.

DISTRIBUTION BIDIMENSIONNELLE ET QUASISTATIONNAIRE DE TEMPERATURE DANS UNE PLAQUE MOBILE INFINIE A PROPRIETES ORTHOTROPES

Résumé—On présente une solution analytique d'un problème avec des conditions aux limites correspondant à une distribution de flux thermique et à une perte par convection. On considère en détail le cas d'une source de chaleur à signal carré sur une face avec des pertes convectives sur les deux faces; des distributions de température sont présentées pour un nombre limité d'exemples et une solution approchée est formulée. Le paramètre conduction/convection, $\chi = \alpha/(Vl)$ fournit un critère pour négliger le transport d'énergie dû à la convection dans la direction du mouvement. Pour $\chi < 0{,}01$, seule la convection est importante. Pour $0{,}01 < \chi < 10$, la conduction et la convection sont toutes deux importantes. La conduction est prépondérante pour $10 < \chi$.

ZWEIDIMENSIONALE QUASI-STATIONÄRE TEMPERATURVERTEILUNG IN EINEM BEWEGLICHEN, UNENDLICH LANGEN STAB MIT ORTHOTROPEN EIGENSCHAFTEN

Zusammenfassung—Es wird eine analytische Lösung des oben genannten Problems mit verteilten Wärmestromdichten und Konvektionsverlusten als Randbedingungen an der Oberfläche entwickelt. Der Fall einer gleichverteilten Wärmestromdichte auf einer Oberfläche mit Konvektionsverlusten auf beiden Oberflächen wird detailliert untersucht; für eine beschränkte Zahl von Beispielen werden die Temperaturverteilungen wiedergegeben und eine Näherungslösung abgeleitet. Der Faktor Wärmeleitung/Konvektion $\chi = \alpha(Vl)$ ist ein Kriterium dafür, ob der Energietransport durch Wärmeleitung oder Konvektion in Bewegungsrichtung vernachlässigt werden kann. Für $\chi \leqslant 0{,}01$ ist nur die Konvektion von Bedeutung. Für $0{,}01 \leqslant \chi \leqslant 10$ spielen Wärmeleitung und Konvektion eine Rolle. Für $10 \leqslant \chi$ ist nur noch die Wärmeleitung wichtig.

ДВУМЕРНОЕ КВАЗИСТАЦИОНАРНОЕ РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ В ДВИЖУЩЕЙСЯ БЕСКОНЕЧНОЙ ПЛИТЕ С ОРТОТРОПНЫМИ ХАРАКТЕРИСТИКАМИ

Аннотация — Разработано аналитическое решение задачи для двумерного квазистационарного распределения температуры в движущейся бесконечной плите с ортотропными характеристиками. Граничные условия на поверхности задаются в виде распределенного теплового потока и конвективных потерь. Подробно рассматривается случай с квадратным импульсным тепловым источником на одной поверхности и конвективными потерями на обеих поверхностях, распределения температуры приводятся для ограниченного числа случаев, и выводится приближенное решение. Показано, что кондуктивно-конвективный параметр $\chi = \alpha/(Vl)$ даёт критерии, благодаря которым можно пренебрегать переносом энергии, вызванным теплопроводностью или конвекцией в направлении движения. Для $\chi \leqslant 0{,}01$ важна только конвекция, для $0{,}01 \leqslant \chi \leqslant 10$ важны как теплопроводность, так и конвекция, а для $10 \leqslant \chi$ — только теплопроводность.